DAY 28

Flying Matrices

Students use a problem context to begin developing the definition of matrix multiplication.

Mathematical Topics

- Finding a quadratic function through specific points by treating the coefficients as variables
- Using the context of a problem to motivate the definition of multiplication of matrices

Outline of the Day

In Class

1. Discuss *Homework 27: Fitting Quadratics*
   - Bring out that "usually" there is a unique parabola through three given points
2. *Flying Matrices*
   - Students use a problem context to begin developing the definition of matrix multiplication
3. Discuss *Flying Matrices*
   - Review that setting up the matrices involves some arbitrary decisions
   - Bring out the arithmetic patterns used in finding the numerical answers
   - Use the arithmetic pattern to define the product of two vectors
   - Discuss how to combine vector products to get a matrix product
   - Save the numerical answers for later reference

At Home

Homework 28: Matrices in the Oven

1. Discussion of *Homework 27: Fitting Quadratics*

The mechanics of these questions are quite similar to students' work on *Homework 14: Fitting a Line* and *Homework 25: Fitting More Lines*.

You can have volunteers present the various parts of Questions 1 through 3. They should come up with this linear system (or equivalent equations).

$$a + b + c = 6$$
$$9a + 3b + c = 8$$
$$25a + 5b + c = 2$$

If many students had trouble finding the three conditions, you may want to give groups some time to work on finding the solution to the system. (The solution is $a = -1$, $b = 5$, $c = 2$.)

Once students find the values for *a, b,* and *c* and combine them to create the quadratic function $y = -x^2 + 5x + 2$, have them verify that it actually fits the three given points. Also discuss the fact that the system of linear equations has a unique solution (Question 4b). Bring out that this illustrates the principle that "usually" there is a unique quadratic function that goes through a given set of three points. You might have students think about what geometric condition on the three points would lead to either no quadratic function or more than one quadratic function through the points.

Note: The only way to get more than one quadratic function is to have two (or all three) of the points be identical. If two of the points have the same x-coordinate but different y-coordinates, then there is no quadratic function through all three points. (In fact, by the definition of *function,* there is no function of any kind through them all.)

If the three points have distinct x-coordinates, then the linear system for the coefficients *a, b,* and *c* will have a unique solution. If the points have distinct x-coordinates and are also collinear, then that unique solution will have $a = 0$, which means that the resulting function is linear rather than quadratic. Therefore, there is no quadratic function through the three points. If the points have distinct x-coordinates and are not collinear, then there is a unique quadratic function that goes through them.

2. *Flying Matrices*

The definition of matrix addition is fairly natural and easy to remember. Matrix multiplication is much more complicated, harder to motivate, and somewhat arbitrary in the way the notation is set up.

The activity *Flying Matrices* motivates the definition of matrix multiplication by having students see the arithmetic involved in the definition through the context of a familiar problem. Tomorrow, students will use this activity and tonight's homework as the basis for the formal definition.

No introductory work is needed for this activity. You should expect students to come up with nonstandard ways to set up these problems. That is okay; tomorrow, they will learn the standard procedure. They should see that it gives essentially the same results as their alternate methods.

You can bring the class together for discussion when all groups have at least finished Questions 1 through 4 and some are finished with the entire activity.

3. Discussion of *Flying Matrices*

For each of Questions 1 through 3 of *Flying Matrices,* let two or three club card students report on how they set up the matrices. As noted previously, you can expect different responses. As you get various answers, record them on overhead transparencies or chart paper so you can refer to them later today and tomorrow.

> Be prepared for a wide variety of methods for setting up matrices, perhaps including methods that are internally inconsistent. Although this variety may create some confusion, it will help students see that some steps they take in the setup process are arbitrary while others need to be done a certain way. As students gain experience, they will see what makes sense.

Here is one *standard* way to set up Questions 1 through 3. (Labels have been added for clarity.)

For Question 1:

$$\begin{array}{c} \\ \text{feed} \\ \text{calcs} \end{array} \begin{array}{c} \begin{array}{cc} \text{wt} & \text{vol} \end{array} \\ \begin{bmatrix} 40 & 2 \\ 50 & 3 \end{bmatrix} \end{array}$$

For Question 2:

$$\begin{array}{c} \\ \text{Mon} \end{array} \begin{array}{c} \begin{array}{cc} \text{feed} & \text{calcs} \end{array} \\ \begin{bmatrix} 500 & 200 \end{bmatrix} \end{array}$$

For Question 3:

$$\begin{array}{c} \\ \text{Mon} \end{array} \begin{array}{c} \begin{array}{cc} \text{wt} & \text{vol} \end{array} \\ \begin{bmatrix} 30{,}000 & 1600 \end{bmatrix} \end{array}$$

But students certainly could have arranged the numbers differently. For example, Question 1 could be set up this way.

$$\begin{array}{c} \\ \text{wt} \\ \text{vol} \end{array} \begin{array}{c} \begin{array}{cc} \text{feed} & \text{calcs} \end{array} \\ \begin{bmatrix} 40 & 50 \\ 2 & 3 \end{bmatrix} \end{array}$$

Also, at this point, the only reason the *feed* column (or row) needs to come before the *calculator* column (or row) is that the information was presented in that order in the problem. As long as the four numbers are included in an organized fashion, the answer should be considered correct at this point. Similarly, the answers to Questions 2 and 3 could be presented as column vectors instead of row vectors.

Presumably, students will all come up with the same numerical totals for Question 3, even if they display the information differently.

• Question 4

The explanations in Question 4 are a crucial part of this assignment, so you should probably have several students describe what they did.

No matter how students displayed the information from Questions 1 through 3, they should have multiplied the appropriate entries of the matrix for Question 2 by the appropriate entries of the matrix for Question 1 and added the products. For example, they should see that the entry "30,000" in the matrix for Question 3 comes from the computation

$$(500 \cdot 40) + (200 \cdot 50)$$

"What do you call a matrix with either only one row or only one column?"

Before proceeding further with the analysis of this computation, you will find it helpful to identify the matrix answer to Question 2 as the special type of matrix called a *vector*. This is either a row vector or a column vector, depending on how each student expressed the answer. We expressed this answer as $\begin{bmatrix} 500 & 200 \end{bmatrix}$.

Similarly, bring out that the two entries comprising the weight information also form either a row or column vector. According to the form we used for Question 2, this information gives the column vector $\begin{bmatrix} 40 \\ 50 \end{bmatrix}$.

"What was done with the two vectors $\begin{bmatrix} 500 & 200 \end{bmatrix}$ and $\begin{bmatrix} 40 \\ 50 \end{bmatrix}$ to produce the entry '30,000' in Question 3?"

Then ask students to describe what has been done with these two vectors to produce the entry "30,000" in Question 3. You want to get an answer something like this: "You multiply the first entry of one by the first entry of the other, then multiply the second entry of one by the second entry of the other, and then add the two products."

Tell students that the result of this process is referred to as the **product of the two vectors.** Point out that this process only makes sense if the two vectors are the same size.

Now move on to Questions 5 and 6. Again, there are likely to be a variety of answers. But you may want to suggest that the answer to Question 5 should be similar to that for Question 2. For example, if a student gave the answer to Question 2 as a 1×2 row vector, then the answer to Question 5 should contain two rows, one for each day, with the first row like the answer from Question 2.

The answers might look like this.

For Question 5:

$$\begin{array}{c} \\ \text{Mon} \\ \text{Tues} \end{array} \begin{array}{c} \begin{array}{cc} \text{feed} & \text{calcs} \end{array} \\ \begin{bmatrix} 500 & 200 \\ 400 & 300 \end{bmatrix} \end{array}$$

For Question 6:

$$\begin{array}{c} \\ \text{Mon} \\ \text{Tues} \end{array} \begin{array}{c} \begin{array}{cc} \text{wt} & \text{vol} \end{array} \\ \begin{bmatrix} 30{,}000 & 1600 \\ 31{,}000 & 1700 \end{bmatrix} \end{array}$$

Again, the key element of the discussion is explaining how the matrix answer to Question 6 comes from the matrix answers for Questions 1 and 5. For example, the entry "1700" represents the volume transported on Tuesday and comes from the computation

$$(400 \cdot 2) + (300 \cdot 3)$$

Using our notation, this can be seen as the product of the *Tuesday row vector* $\begin{bmatrix} 400 & 300 \end{bmatrix}$ (from the matrix of Question 5) and the *volume column vector*

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

(from the matrix of Question 1).

Be sure students know what each of the numbers in the answer to Question 6 represents. For example, they should be able to express that the entry "31,000" represents the weight of Linda Sue's load on Tuesday.

You can tell students at this point that the matrix that results from combining the answers for Questions 1 and 5 in this way is called the **product** of the matrices. Tell them further that they will be seeing a standard way to set up the matrices so that this multiplication can be done routinely on calculators or computers.

Homework 28: Matrices in the Oven

Tonight's homework is similar to the *Flying Matrices* activity.